$$\underset{\swarrow\, n\times p}{} r = \underline{r(X) < p} \quad + \text{ conditions } \quad \Rightarrow \quad \underline{\text{unique solution of } \beta}$$
$$p - r = s$$

† Imposing conditions for a unique solution to the NEs.

$$(X^T X)\beta = X^T y$$

♠ Ex2 (Contd): Consider the one-way ANOVA model.

$$y_{ij} = \mu + \alpha_i + e_{ij},$$

where $i = 1, 2, 3;\ j = 1, 2.$

$$y = X\beta + e$$

$$\begin{bmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \\ y_{31} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{6\times 4} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} e_{11} \\ \vdots \\ e_{32} \end{bmatrix}$$

$r(X) = 3$

$$\underset{4\times4}{X^T X} = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \qquad X^T y = \begin{bmatrix} \Sigma y_{ij} \\ \Sigma y_{1j} \\ \Sigma y_{2j} \\ \Sigma y_{3j} \end{bmatrix} = \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} \\ y_{2\cdot} \\ y_{3\cdot} \end{bmatrix}$$

$$\begin{aligned} 6\mu + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 &= y_{\cdot\cdot} \\ 2\mu + 2\alpha_1 &= y_{1\cdot} \\ 2\mu + 2\alpha_2 &= y_{2\cdot} \\ 2\mu + 2\alpha_3 &= y_{3\cdot} \end{aligned}$$

① $\sum n_i \alpha_i = 0$ ($\Rightarrow$ balanced $\sum \alpha_i = 0$)

$0 + \alpha_1 + \alpha_2 + \alpha_3 = 0$

$6\mu = y_{\cdot\cdot}$

$\Rightarrow \hat{\mu} = \bar{y}_{\cdot\cdot} = \frac{y_{\cdot\cdot}}{6}$

$\Rightarrow \hat{\alpha}_i = \frac{y_{i\cdot}}{2} - \hat{\mu}$

$= \bar{y}_{i\cdot} - \hat{\mu}$

② $\alpha_1 = 0$

$0 + \alpha_1 + 0 + 0 = 0$

$\hat{\mu} = \bar{y}_{1\cdot}$

$\hat{\alpha}_i = \bar{y}_{i\cdot} - \bar{y}_{1\cdot}, \quad i = 2, 3$

$= \bar{y}_{i\cdot} - \hat{\mu}$

③ $\mu = 0$

$\hat{\mu} = 0$

$\hat{\alpha}_i = \bar{y}_{i\cdot}$

$s = p - r$

consider equations of the form $\underset{s\times p}{C}\beta = 0$

† Imposing conditions for a unique solution to the NEs.

- We add rows to $\mathbf{X}$ to make $\mathbf{X}$ full-column rank (rank=p).

$$\underbrace{\begin{bmatrix} \mathbf{X}^T\mathbf{X} \\ \mathbf{C} \end{bmatrix}}_{(p+s)\times \cancel{(p+s)}\, p} \beta = \begin{bmatrix} \mathbf{X}^T\mathbf{y} \\ \mathbf{0} \end{bmatrix}$$

$\mathbf{C}$: $s \times p$ matrix with $s = p - r$ and $C([\mathbf{X}^T \;\; \mathbf{C}^T]) = \mathbb{R}^p$.

† Question: How to choose such $\mathbf{C}$?

$s = p - r$

Columns of $\mathbf{C}^T$ must have some contribution from the basis vectors of $\mathcal{N}(\mathbf{X})$. $C^T \in \mathcal{C}(X^T)$ $[X^T \; C^T]$

In other words,

- The columns of $\mathbf{C}^T$ cannot be orthogonal to $\mathcal{N}(\mathbf{X})$. $\perp \mathcal{C}(X^T)$
- The components of $\mathbf{C}\boldsymbol{\beta}$ are nonestimable.

♠ Ex3: Two-Way Crossed without Interaction

Consider the two-way crossed model without interaction:

$$y_{ij} = \mu + \alpha_i + \beta_j + e_{ij}, \qquad i = 1, 2, \text{ and } j = 1, 2, 3,$$

so $N = 6$ and $p = 6$.

1. What is $r = rank(\mathbf{X})$?
2. Write out the normal equations.
3. Give a set of basis vectors of $\mathcal{N}(\mathbf{X})$.
4. Give a list of r linearly independent functions $\boldsymbol{\lambda}^T\boldsymbol{\beta}$.
5. Find $\mathbf{C}^T$ for a unique solution and find the corresponding solution.
6. Show that $\alpha_1 - \alpha_2$ is estimable and give its least squares estimator.
7. Show that $\beta_1 - 2\beta_2 + \beta_3$ is estimable and gives its least square estimator.

$y_{ij} = \mu + \alpha_i + \beta_j + e_{ij} \qquad \underset{(=a)}{i=1,2} \qquad \underset{(=b)}{j=1,2,3}$

$$\begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}}_{\boxed{6\times 6}} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{12} \\ \vdots \\ e_{23} \end{bmatrix}$$

1 $\quad r(X) = r(X^T X) = 4 = a+b-1$

2. $\quad X^T X \beta = X^T y$

$\cancel{X^T X = \begin{bmatrix} 1_6^T & \\ 1_3^T & 0_3 \end{bmatrix}}$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 & 3 & 3 & 2 & 2 & 2 \\ 3 & 3 & 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 & 0 & 2 \end{bmatrix}}_{r(X^T X)=4}$$

$$X^T y = \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} \\ y_{2\cdot} \\ y_{\cdot 1} \\ y_{\cdot 2} \\ y_{\cdot 3} \end{bmatrix}$$

$$\begin{aligned}
6\mu + 3\alpha_1 + 3\alpha_2 + 2\beta_1 + 2\beta_2 + 2\beta_3 &= y_{\cdot\cdot} \\
3\mu + 3\alpha_1 + \beta_1 + \beta_2 + \beta_3 &= y_{1\cdot} \\
3\mu + 3\alpha_2 + \beta_1 + \beta_2 + \beta_3 &= y_{2\cdot} \\
2\mu + \alpha_1 + \alpha_2 + 2\beta_1 &= y_{\cdot 1} \\
2\mu + \alpha_1 + \alpha_2 + 2\beta_2 &= y_{\cdot 2} \\
2\mu + \alpha_1 + \alpha_2 + 2\beta_3 &= y_{\cdot 3} \\
\alpha_1 + \alpha_2 &= 0 \\
\beta_1 + \beta_2 + \beta_3 &= 0
\end{aligned}$$

$\hat{\mu} = \dfrac{y_{\cdot\cdot}}{6} = \bar{y}_{\cdot\cdot}$

$\hat{\alpha}_1 = \bar{y}_{1\cdot} - \hat{\mu}$

$\hat{\alpha}_2 = \bar{y}_{2\cdot} - \hat{\mu}$

$\hat{\beta}_j = \bar{y}_{\cdot j} - \hat{\mu}, \quad j = 1, 2, 3$

3. $N(X) = \{\lambda \mid X\lambda = 0\} = N(X^T X)$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

basis for $N(X) = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix} \right\}$

4. $\mathcal{C}(X^T)$ Find a basis of $\mathcal{C}(X^T)$

$$X^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

a basis for $\mathcal{C}(X^T) = \left\{ \underset{u_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{u_4}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$

$\underline{\underline{u_j^T \beta}}$: linearly independent estimable functions

5. <u>Usual</u> conditions: $\sum \alpha_i = 0$ & $\sum \beta_j = 0$

$$C^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \leftarrow c_1^T \\ \leftarrow c_2^T \end{matrix}$$

$c_1, c_2 \notin \mathcal{C}(X^T)$

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathcal{C}(X^T)} \quad \underbrace{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}}_{N(X)}$$

$\quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad v_1 \quad v_2$

$c_1 \notin \mathcal{C}(X^T)$

$$c_1 = 2u_1 + 1u_2 + 1u_3 + 2u_4 - 8v_1 + 2v_2$$

$$c_2 = 2u_1 + 1u_2 + 1u_3 + 2u_4 + 3v_1 - 9v_2$$

$$\left.\begin{matrix} c_1 \perp v_1 \times \\ c_2 \perp v_2 \times \end{matrix}\right] \Rightarrow c_1 \notin \mathcal{C}(X^T)$$

Otherwise

$r([X^T\ C^T]) = p = 6.$

6. $\alpha_1 - \alpha_2$: Yes estimable!!

1. $E(y_{11}) - E(y_{21}) = (\mu + \alpha_1 + \beta_1) - (\mu + \alpha_2 + \beta_1) = \underline{\alpha_1 - \alpha_2}$

2. $\lambda = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

3. $\lambda \perp V_1$ $\quad \left.\begin{array}{l} \lambda^T V_1 = 0 \\ \lambda^T V_2 = 0 \end{array}\right)$ $\quad \lambda \perp N(X)$

$\Rightarrow \lambda \in C(X^T)$

$\Rightarrow \alpha_1 - \alpha_2$: estimable

$$\lambda^T \hat{\beta} = \hat{\alpha}_1 - \hat{\alpha}_2 = \underbrace{(\bar{y}_{1\cdot} - \hat{\mu})} - \underbrace{(\bar{y}_{2\cdot} - \hat{\mu})} = \bar{y}_{1\cdot} - \bar{y}_{2\cdot}$$

7. $\beta_1 - 2\beta_2 + \beta_3$

1. $E(y_{11}) - 2E(y_{12}) + E(y_{13}) = \beta_1 - 2\beta_2 + \beta_3$

2. $\lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

3. $\left.\begin{array}{l} \lambda \perp V_1 \\ \lambda \perp V_2 \end{array}\right) \Rightarrow \lambda \in C(X^T) \Rightarrow \beta_1 - 2\beta_2 + \beta_3$ estimable.

$$\lambda^T \hat{\beta} = \hat{\beta}_1 - 2\hat{\beta}_2 + \hat{\beta}_3 = \bar{y}_{\cdot 1} - 2\bar{y}_{\cdot 2} + \bar{y}_{\cdot 3}$$

- HW 2 will be on website
- HW 1 solution will be
- ~~4/21 (Th)~~ : test 1
  or 26 (Tue) ?

Gauss-Markov Model &

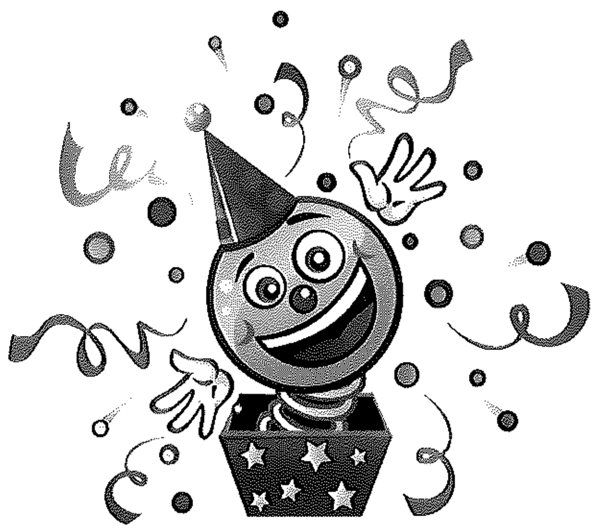
Today: Finish Chapter 3 & Start chapter 4 (GLS)

imposing constraints

AMS 256
Chapter 4: Gauss-Markov Model

Spring 2016

- More linear algebra!



- Consider $\mathbf{A}$ a $n \times n$ matrix.

† Def: The *trace* of $\mathbf{A}$ is a scalar given by the sum of the diagonal elements of $\mathbf{A}$, that is, $tr(\mathbf{A}) = \sum_i a_{ii}$.

★ Result:

- $tr(\mathbf{AB}) = tr(\mathbf{BA})$
- $tr(\mathbf{A}^T\mathbf{A}) = \sum_{i,j} a_{ij}^2$

$$\frac{1}{(\sqrt{2\pi})^n |\Sigma|^{1/2}}$$

† Def: The *determinant* of an $n \times n$ matrix $\mathbf{A}$ is a scalar given by

$$|\mathbf{A}| = \sum_{j=1} a_{ij}(-1)^{i+j}|\mathbf{M}_{ij}|, \text{ for any fixed } i, \text{ or}$$

$$|\mathbf{A}| = \sum_{i=1} a_{ij}(-1)^{i+j}|\mathbf{M}_{ij}|, \text{ for any fixed } j,$$

where $|\mathbf{M}_{ij}|$ (minor corresponding to a_{ij}) is the determinant of the $(n-1) \times (n-1)$ submatrix of $\mathbf{A}$ after deleting row i and column j from $\mathbf{A}$.

★ Result A.17:

- $det(\mathbf{A}) = |\mathbf{A}| = a_{11}a_{22} - a_{23}a_{32}$ where $\mathbf{A}$ is a 2×2 square matrix
- $|\mathbf{D}_n| = \prod_i d_{ii}$ where $\mathbf{D}$ is a diagonal matrix
- $|\mathbf{AB}| = |\mathbf{BA}|$, $\quad |\mathbf{A}^{-1}| = 1/|\mathbf{A}|$, $\quad |c\mathbf{A}| = c^n|\mathbf{A}|$

- Recall!

★ Result: The following is equivalent; for $\mathbf{A}$ a $n \times n$ matrix,

- $\mathbf{Ax} = \mathbf{0}$ implies $\mathbf{x} = 0$ (that is, $r(\mathbf{A}) = n$ and $\mathbf{A}$ is non singular)
- $|\mathbf{A}| \neq 0$.
- $\mathbf{A}$ is invertible (that is, $\exists \ \mathbf{A}^{-1}$)

★ Let **A** a $n \times n$ square and **symmetric** matrix. Consider an equation;

$$\mathbf{Aq} = \lambda \mathbf{q} \tag{1}$$

where λ is the eigenvalue and $\mathbf{q}(\neq \mathbf{0})$ is its associated eigenvector.

- Results:
  - There are n real eigenvalues (roots to (1)), λ_j, $j = 1, \ldots, n$ (λ_j can be zero or a multiple root).
  - Eigenvectors associated wtih distinct eigenvalues are orthogonal. $\lambda_j \neq \lambda_{j'} \Rightarrow q_j \perp q_{j'} = 0$
  - For eigenvalues with multiplicity, eigenvectors can be chosen to be orthogonal.
  - Eigenvectors can be normalized ($\mathbf{q}_j^T \mathbf{q}_j = 1$)

$$A = \begin{bmatrix} 10 & 3 & 2 \\ 3 & 9 & 3 \\ 2 & 3 & 10 \end{bmatrix} \qquad tr(A) = 10 + 9 + 10 = \underline{\underline{29}}$$

$$= 15 + 8 + 6$$

$$|A| = 10(-1)^{1+1}\begin{vmatrix} 9 & 3 \\ 3 & 10 \end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix} 3 & 3 \\ 2 & 10 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 3 & 9 \\ 2 & 3 \end{vmatrix}$$

$$= \underline{\underline{720}} = 15 \times 8 \times 6$$

$$Aq = \lambda q \quad \Rightarrow \quad (A - \lambda I)q = 0$$

$$\Rightarrow \quad |A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 10-\lambda & 3 & 2 \\ 3 & 9-\lambda & 3 \\ 2 & 3 & 10-\lambda \end{bmatrix}$$

$$\Rightarrow \quad \lambda_1 = 15, \quad \lambda_2 = 8, \quad \lambda_3 = 6$$

$$\Rightarrow \quad q_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad q_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$A^{-1} = Q \begin{bmatrix} \frac{1}{15} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} Q^T$$

$$A^{1/2} = Q \begin{bmatrix} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{8} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix} Q^T$$

$$A^{-1/2} = Q \begin{bmatrix} \frac{1}{\sqrt{15}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{8}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{bmatrix} Q^T$$

★ Results The *spectral decomposition* of a symmetric matrix $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T = \sum_j \lambda_j \mathbf{q}_j \mathbf{q}_j^T.$$

- $|\mathbf{A}| = \prod_j \lambda_j$ $\quad \lambda = 0 \quad \cancel{A = \lambda} \quad Aq = \lambda q$
- $tr(\mathbf{A}) = \sum_j \lambda_j$ $\quad \cancel{(A - \lambda I)q = 0}$
- $r(\mathbf{A})$ = the number of nonzero eigenvalues $\quad Aq = 0$
- $\mathbf{A}^{-1} = \mathbf{Q}\Lambda^{-1}\mathbf{Q}^T$ $\quad \text{diag}(\frac{1}{\lambda_j})$
- The square-root matrix of a nonnegative definite matrix

$$\mathbf{A}^{1/2} = \mathbf{Q}\Lambda^{1/2}\mathbf{Q}^T$$

$(\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$ and $\underbrace{(\mathbf{A}^{1/2})^T = \mathbf{A}^{1/2}}_{\text{symmetric}})$

- From the previous slide, $\mathbf{Aq} = \lambda\mathbf{q}$.

- Let's stack the eigenvectors of $\mathbf{A}$ as columns of a matrix $\mathbf{Q}$. Let Λ a diagonal matrix of the eigenvalues, ordered in the same way the eigenvectors are stacked in $\mathbf{Q}$. Then we get

$$\mathbf{AQ} = \mathbf{Q}\Lambda$$

- Since $\mathbf{q}_j$ are mutually orthogonal, we can easily check $\mathbf{Q}^{-1} = \mathbf{Q}^T$ and $\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = I_n$.

- We obtain the *spectral decomposition* of a symmetric matrix,

$$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T = \sum_j \lambda_j \mathbf{q}_j \mathbf{q}_j^T.$$

• Let's connect the spectral decomposition of a symmetric matrix $\mathbf{A}$ into a basis of $C(\mathbf{A})$ and $\mathcal{N}(\mathbf{A}^T) = \mathcal{N}(\mathbf{A})$.

† Th: If $\mathbf{A}$ is a symmetric matrix, then there exists an orthonormal basis of $C(\mathbf{A})$ consisting of eigenvectors of nonzero eigenvalues. If λ is a nonzero eigenvalues of multiplicity s, then the basis will contain s eigenvectors for λ.

$\Rightarrow$ What does this mean?

$\Rightarrow$ Well, it means... For $n \times n$ symmetric matrix $\mathbf{A}$,

- The total number of eigenvalues is n. If a particular λ is an eigenvalue with multiplicity s, then we can think this as λ appears s times (still those s eigenvectors corresponding to λ can be chosen to be mutually orthogonal).

- $\mathbf{q}_j$, $j = 1, \ldots, n$ is an orthonormal basis of eigenvectors for $\mathbb{R}^n$, with $\mathbf{q}_j$ being an eigenvector for λ_j for any j.

- $r(\mathbf{A}) = \#$ of nonzero eigenvalues.

- Eigenvectors of $\overset{= \lambda_j}{0}$ are the null space of $\mathbf{A}$.

- For $\mathbf{A}$ symmetric, $C(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ are orthogonal compliments.

- Eigenvectors of nonzero λ_j are an orthonormal basis of $C(\mathbf{A})$.

- Eigenvectors of zero λ_j are an orthonormal basis of $\mathcal{N}(\mathbf{A})$.

• Let's connect the spectral decomposition of a symmetric matrix $\mathbf{A}$ into $\mathbf{A}^{-}$.

$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\Lambda^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

From the previous slide,

$$\mathbf{A}^{-1} = \mathbf{Q}\Lambda^{-1}\mathbf{Q}^{T}$$

where Λ is the diagonal matrix with λ_j.

† Th: Let $\mathbf{A}$ be a $n \times n$ symmetric matrix of rank r ($r \leq n$). Let Λ_r be the diagonal matrix containing its nonzero eigenvalues (in decreasing order of magnitude), and let $\mathbf{Q}_r$ be the $n \times r$ matrix whose columns are the eigenvectors corresponding to the nonzero λ_j of $\mathbf{A}$. Then,

$$\mathbf{A}^{-} = \mathbf{Q}_r\Lambda_r^{-1}\mathbf{Q}_r^{T}.$$

Or this is the same as letting Λ^{-1} with $\underline{1/\lambda_j}$ for $\lambda_j \neq 0$ and 0 for $\lambda_j = 0$.

$$-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)$$

♣ Def: A *quadratic form* in the n variables is $\mathbf{x}^T\mathbf{A}\mathbf{x}$ where n–dim vector $\mathbf{x}$ and $\mathbf{A}$ is a $n \times n$ symmetric matrix.

Observe

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{i,j} a_{ij} x_i x_j,$$

that is, it has only squared terms x_i^2 and product terms, $x_i x_j$.

† Th: A symmetric matrix $\mathbf{A}$ is *positive* (*nonnegative*) *definite* if, for nay nonzero vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v}^T\mathbf{A}\mathbf{v}$ is positive (nonnegative).

$$\begin{aligned} 0 &\leq \mathbf{x}^T\mathbf{A}\mathbf{x} && \text{for all } \mathbf{x}, \\ 0 &< \mathbf{x}^T\mathbf{A}\mathbf{x} && \text{for all } \mathbf{x}(\neq \mathbf{0}). \end{aligned}$$

- Let's connect the spectral decomposition of a symmetric matrix $\mathbf{A}$ into definiteness.

★ Results:

- a $n \times n$ symmetric matrix $\mathbf{A}$ is positive definite iff every eigenvalue of $\mathbf{A}$ is positive
- a $n \times n$ symmetric matrix $\mathbf{A}$ is nonnegative definite iff all of its eigenvalues are greater than or equal to zero
- nonnegative definite + nonsingular $\Rightarrow$ positive definite

★ Th: $\mathbf{A}$ is nonnegative definite iff there exists a square matrix $\mathbf{L}$ such that $\mathbf{A} = \mathbf{L}\mathbf{L}^T$.

★ Th: $\mathbf{A}$ is positive definite iff $\mathbf{L}$ is nonsingular for any choice $\mathbf{L}$.

$\Rightarrow$ how to find such $\mathbf{L}$?

$\surd$ Cholesky factorization

$\surd$ Spectral decomposition

★ Th: A square matrix **A** is positive definite iff there exits a nonsingular lower triangular matrix **L** such that $\mathbf{A} = \mathbf{L}\mathbf{L}^T$.

Finding such nonsingular lower triangular matrix **L** can be done by Cholesky factorization (check A.7 of M for more).

• Let **A** a $n \times n$ symmetric nonnegative matrix, the eigvenvalues are nonnegative and let $\Lambda^{1/2}$ be the diagonal matrix with $\sqrt{\lambda_i}$ (if $\lambda_j > 0$) and 0 (if $\lambda_j = 0$). We know

⋆ the symmetric square root matrix of **A**, $\mathbf{A}^{1/2} = \mathbf{Q}\Lambda^{1/2}\mathbf{Q}^T$

⋆ The square of $\mathbf{A}^{1/2}$ produces **A**.

$$A^{1/2} A^{1/2} = Q \Lambda^{1/2} \underbrace{Q^T Q}_{I} \Lambda^{1/2} Q^T$$

$$= Q \Lambda^{1/2} \Lambda^{1/2} Q^T$$

$$= Q \Lambda Q^T$$

$$= A$$