Spring 16 – AMS256 Homework 4

1. Consider the following two-way ANOVA model:

$$y_{i,j,k} = \mu + \alpha_i + \beta_j + \epsilon_{i,j,k}.$$

Assume that the response values are given in the following table:

		Factor B		
		1	2	3
Factor A	1	17, 20	15	20
	2	12		11, 14
	3	6		17
	4	9	4, 6	19

Note that empty cells mean no observation. Show that $H_0 : \beta_1 = \beta_2 = \beta_3$ is testable and test H_0 at the 5% level.

2. Let

$$\begin{aligned} y_1 &= \alpha_1 + \epsilon_1, \\ y_2 &= 2\alpha_2 - \alpha_2 + \epsilon_2, \\ y_3 &= \alpha_1 + 2\alpha_2 + \epsilon_2, \end{aligned}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)^T \sim \mathrm{N}_3(0, \sigma^2 I)$. Derive the F-statistic for testing $H_0 : \alpha_1 = \alpha_2$.

3. Consider the model

$$\begin{aligned} \mathrm{E}(y_i) &= \beta_{1,0} + \beta_{1,1} x_i, \quad i = 1, \ldots, n, \\ \mathrm{E}(y_i) &= \beta_{2,0} + \beta_{2,1} x_i, \quad i = n+1, \ldots, n+m. \end{aligned}$$

Suppose y_i's are independent normal random variables with variance σ^2. Let γ denote the value of x at which the lines intersect. Find the MLEs of $\beta_{1,0}$, $\beta_{1,1}$, $\beta_{2,0}$, $\beta_{2,1}$, σ^2 and γ. Construct a 95% confidence interval for γ. Does such an interval always exist?

4. Consider the balanced one-way ANOVA model. Show that if $\boldsymbol{c}_i^T \hat{\beta}$ and $\boldsymbol{c}_j^T \hat{\beta}$ are orthogonal contrasts (i.e., $\boldsymbol{c}_i^T \boldsymbol{c}_j = 0$ for $i \neq j$), then $\boldsymbol{c}_i^T \hat{\beta}$ and $\boldsymbol{c}_j^T \hat{\beta}$ are independent.

5. Consider the one-way ANOVA model

$$y_{i,j} = \mu + \alpha_i + \epsilon_{i,j} \equiv \mu_i + \epsilon_{i,j},$$

with $\epsilon_{i,j}$ i.i.d. $\mathrm{N}(0,1)$ for $i = 1 : k$, and $j = 1 : n_i$. Show that two contrasts $\hat{\delta} = \sum_{i=1}^k a_i \bar{y}_{i,\cdot}$, and $\hat{\gamma} = \sum_{i=1}^k b_i \bar{y}_{i,\cdot}$, are independent if and only if $\sum_{i=1}^k a_i b_i / n_i = 0$.

6. Let $\boldsymbol{P} = \boldsymbol{X}(\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T$. Verify the following properties of the residual vector $\hat{\boldsymbol{\epsilon}}$:

   (a) $\mathrm{E}(\hat{\boldsymbol{\epsilon}}) = \boldsymbol{0}$

   (b) $\mathrm{Cov}(\hat{\boldsymbol{\epsilon}}) = \sigma^2(\boldsymbol{I} - \boldsymbol{P})$.

   (c) $\mathrm{Cov}(\hat{\boldsymbol{\epsilon}}, \boldsymbol{y}) = \sigma^2(\boldsymbol{I} - \boldsymbol{P})$.

   (d) $\mathrm{Cov}(\hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{y}}) = \boldsymbol{0}$.