(a) i. $\Lambda = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow$ rank$(\Lambda) = 2$ (columns are linearly indep) ... (A)

$\qquad\quad \overset{\shortparallel}{\lambda_1} \quad \overset{\shortparallel}{\lambda_2}$

$X = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

Basis vectors for $\mathcal{C}(X^T)$:

$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\quad = u_1 \qquad = u_2 \qquad = u_3 \qquad = u_4$

$\Rightarrow$ rank$(X) = 4$

(B) $\begin{cases} \lambda_1 = u_1 - u_2 \Rightarrow \lambda_1 \in \mathcal{C}(X^T) \Rightarrow \lambda_1^T\beta \text{ estimable} \\ \lambda_2 = u_3 - u_4 \Rightarrow \lambda_2 \in \mathcal{C}(X^T) \Rightarrow \lambda_2^T\beta \text{ estimable} \end{cases}$

By (A) & (B) $\Lambda^T\beta = 0$ is testable.

(b) From lecture, BLUE of $\Lambda^T\beta$ is $\Lambda^T\hat{\beta}$ where $\hat{\beta}$ : LSE

$\Lambda^T\hat{\beta} = \Lambda^T(X^TX)^- X^T y \sim N_2(\underbrace{\Lambda^T\beta}_{\text{unbiased}}, \sigma^2\Lambda^T(X^TX)^-\Lambda)$ : $y$ follows a n-dim normal ($n=8$) $\Rightarrow$ linear combinations of $y$ follow a normal

Given in the question

$\hat{\beta} = (X^TX)^- X^Ty = \begin{bmatrix} \bar{y}_{\cdot\cdot} \\ \bar{y}_{1\cdot} - \bar{y}_{\cdot\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{\cdot\cdot} \\ \bar{y}_{3\cdot} - \bar{y}_{\cdot\cdot} \\ \bar{y}_{4\cdot} - \bar{y}_{\cdot\cdot} \end{bmatrix} \Rightarrow \Lambda^T\hat{\beta} = \begin{bmatrix} \bar{y}_{1\cdot} - \bar{y}_{2\cdot} \\ \bar{y}_{3\cdot} - \bar{y}_{4\cdot} \end{bmatrix}$

$$\sigma^2\Lambda^T(X^TX)^-\Lambda = \sigma^2 \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \Lambda^T \hat{\beta} = \begin{bmatrix} \bar{y}_{1\cdot} - \bar{y}_{2\cdot} \\ \bar{y}_{3\cdot} - \bar{y}_{4\cdot} \end{bmatrix} \sim N_2 \left( \begin{bmatrix} \alpha_1 - \alpha_2 \\ \alpha_3 - \alpha_4 \end{bmatrix}, \ \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

(c). (i) $\text{rank}(X) = 4$

(iii) $\dfrac{SSE}{\sigma^2} = \dfrac{y^T(I-P)y}{\sigma^2} \sim \chi^2(8-4)$

TRUE

$$\Lambda^T \hat{\beta} = \Lambda^T (X^T X)^- X^T y = (X^T A)^T (X^T X)^- X^T y = A^T X (X^T X)^- X^T y$$
$$= A^T P y$$

since $\lambda_1, \lambda_2 \in \mathcal{C}(X^T)$

$\exists a_1, a_2$ s.t $\lambda_1 = X^T a_1$ & $\lambda_2 = X^T a_2$

Let $A = [a_1 | a_2] \Rightarrow X^T A = \Lambda$

$$\text{cov}((I-P)y, A^T P y) = (I-P)\,\text{cov}(y, y)\, P A$$
$$= \sigma^2 (I-P) P A$$
$$= 0$$

$\Rightarrow$ since $(I-P)y$ and $A^T P y$ are normal,

$\text{cov}((I-P)y, A^T P y) = 0 \Leftrightarrow (I-P)y$ & $A^T P y$ are indep.

$\left.\begin{array}{l} \dfrac{SSE}{\sigma^2} \text{ is a function of } (I-P)y \\ \Lambda^T \hat{\beta} \text{ is a function of } A^T P y \end{array}\right) \Rightarrow \dfrac{SSE}{\sigma^2}$ & $\Lambda^T \hat{\beta}$ are indep.

(d)

$$\Lambda^T\hat{\beta} \sim N_2(\Lambda^T\beta,\ \sigma^2 I_2)$$

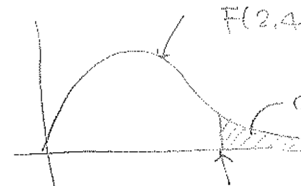
$$\Rightarrow \left[\begin{array}{l} \frac{1}{\sigma^2}(\Lambda^T\hat{\beta})^T(\Lambda^T\hat{\beta}) \sim \chi^2\left(2, \frac{(\Lambda^T\beta)^T(\Lambda^T\beta)}{2}\right) \\ \frac{SSE}{\sigma^2} \sim \chi^2(4) \end{array}\right] \text{ indep.}$$

$$F = \frac{(\Lambda^T\hat{\beta})^T(\Lambda^T\hat{\beta})/2}{SSE/4} \sim F\left(2, 4, \underbrace{\frac{(\Lambda^T\beta)^T(\Lambda^T\beta)}{2}}_{}\right) \quad =\phi$$

$$\Rightarrow F = \frac{\left\{(\bar{y}_{1\cdot} - \bar{y}_{2\cdot})^2 + (\bar{y}_{3\cdot} - \bar{y}_{4\cdot})^2\right\}/2}{\sum_{i=1}^{4}\sum_{j=1}^{2}(y_{ij} - \bar{y}_{i\cdot})^2/4} \sim F\left(2, 4, \underbrace{\frac{(\alpha_1-\alpha_2)^2 + (\alpha_3-\alpha_4)^2}{2}}_{\substack{=0 \\ \text{under } H_0}}\right)$$

The test procedure is

Reject $H_0: \alpha_1 - \alpha_2 = 0\ \&\ \alpha_3 - \alpha_4 = 0$ if $F > F_\alpha(2,4)$




V. As $|\alpha_1 - \alpha_2|$ becomes bigger OR $|\alpha_3 - \alpha_4|$ becomes bigger

$(\alpha_1-\alpha_2)^2$ and/or $(\alpha_3-\alpha_4)^2$ increases. $\Rightarrow$ the noncentrality parameter ($\phi$) is increasing.

Since the noncentral F distribution is stocastically increasing in $\phi$, the power of the developed test is increasing.

power $= P(F > F_\alpha(2,4) \mid H_0 \text{ is not true})$

2 (a) $X_0 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ , $\beta_0 = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$

$\text{rank}(X_0) = 2$ .

(b) $P_0 = X_0 (X_0^T X_0)^{-} X_0^T =$

$$X_0^T X_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 4 \\ 4 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow (X_0^T X_0)^{-} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$(X_0^T X_0)^{-} X_0^T = \underset{3 \times 3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}} \underset{3 \times 8}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$Py = \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \\ \bar{y}_{2\cdot} \\ \bar{y}_{3\cdot} \\ \bar{y}_{3\cdot} \\ \bar{y}_{4\cdot} \\ \bar{y}_{4\cdot} \end{bmatrix} \quad P_0 y = \begin{bmatrix} \frac{\bar{y}_{1\cdot}+\bar{y}_{2\cdot}}{2} \\ \frac{\bar{y}_{1\cdot}+\bar{y}_{2\cdot}}{2} \\ \frac{\bar{y}_{1\cdot}+\bar{y}_{2\cdot}}{2} \\ \frac{\bar{y}_{3\cdot}+\bar{y}_{2\cdot}}{2} \\ \frac{\bar{y}_{3\cdot}+\bar{y}_{4\cdot}}{2} \\ \frac{\bar{y}_{3\cdot}+\bar{y}_{4\cdot}}{2} \\ \frac{\bar{y}_{3\cdot}+\bar{y}_{4\cdot}}{2} \\ \frac{\bar{y}_{3\cdot}+\bar{y}_{4\cdot}}{2} \end{bmatrix} \Rightarrow (P-P_0)y = \begin{bmatrix} \frac{\bar{y}_{1\cdot}-\bar{y}_{2\cdot}}{2} \\ \frac{\bar{y}_{1\cdot}-\bar{y}_{2\cdot}}{2} \\ \frac{\bar{y}_{2\cdot}-\bar{y}_{1\cdot}}{2} \\ \frac{\bar{y}_{2\cdot}-\bar{y}_{1\cdot}}{2} \\ \frac{\bar{y}_{3\cdot}-\bar{y}_{4\cdot}}{2} \\ \frac{\bar{y}_{3\cdot}-\bar{y}_{4\cdot}}{2} \\ \frac{\bar{y}_{4\cdot}-\bar{y}_{3\cdot}}{2} \\ \frac{\bar{y}_{4\cdot}-\bar{y}_{3\cdot}}{2} \end{bmatrix}$$

$$\Rightarrow \frac{y^T(P-P_0)y}{\sigma^2} = \frac{4(\bar{y}_{1\cdot}-\bar{y}_{2\cdot})^2 + 4(\bar{y}_{3\cdot}-\bar{y}_{4\cdot})^2}{4\sigma^2} = \frac{(\bar{y}_{1\cdot}-\bar{y}_{2\cdot})^2 + (\bar{y}_{3\cdot}-\bar{y}_{4\cdot})^2}{\sigma^2}$$

$\frac{y}{\sigma} \sim N_n(X\beta, I_n)$ & $(P-P_0)$ is symmetric & idempotent

$$\Rightarrow \frac{y^T(P-P_0)y}{\sigma^2} \sim \chi^2\left( \underset{\substack{\| \\ 4-2 \\ =2}}{r(P-P_0)}, \frac{(X\beta)^T(P-P_0)X\beta}{2\sigma^2} \right)$$

by the results in lecture

(c) TRUE $\quad Cov((P-P_0)y, (I-P)y) = \sigma^2(P-P_0)(I-P)$

$$= \sigma^2(P - P^2 - P_0 - P_0P)$$

$$= \sigma^2(P - P - P_0 - P_0) \quad \text{since } P \text{ is idempotent}$$

$$= 0 \quad \text{and } \mathcal{C}(X_0) \subset \mathcal{C}(X)$$

$\Rightarrow$ since $(P-P_0)y$ and $(I-P)y$ are normal,

$Cov = 0 \iff (P-P_0)y$ and $(I-P)y$ are indep.

(d) $\boxed{\text{Test statistic}}$ since $\frac{y^T(P-P_0)y}{\sigma^2}$ and $\frac{SSE}{\sigma^2}$ are indep,

$$F = \frac{\dfrac{y^T(P-P_0)y}{\cancel{\sigma^2}} \Big/ 2}{\dfrac{SSE}{\cancel{\sigma^2}} \Big/ 4} = \frac{2\cdot\left( (\bar{y}_{1\cdot}-\bar{y}_{2\cdot})^2 + (\bar{y}_{3\cdot}-\bar{y}_{4\cdot})^2 \right)}{\sum_i\sum_j (y_{ij}-\bar{y}_{i\cdot})^2}$$

$$\sim F\Bigg(2, 4, \underbrace{\frac{(X\beta)^T(P-P_0)(X\beta)}{2\sigma^2}}_{\phi}\Bigg)$$

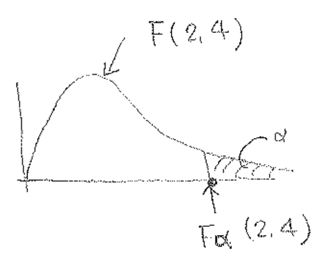
since $C(X_0) \subset C(X)$, $PX_0\beta_0 = P_0X_0\beta_0$

under $H_0$, $E((P-P_0)y) = (P-P_0)X_0\beta_0 = 0 \Rightarrow \phi = 0$

under $H_0$ not true, $E((P-P_0)y) = (P-P_0)X\beta = PX\beta - P_0X\beta = X\beta - P_0X\beta$

$$= \begin{bmatrix} \mu+\alpha_1 \\ \mu+\alpha_1 \\ \vdots \\ \mu+\alpha_4 \\ \mu+\alpha_4 \end{bmatrix} - \begin{bmatrix} \mu+\frac{\alpha_1+\alpha_2}{2} \\ \mu+\frac{\alpha_1+\alpha_2}{2} \\ \vdots \\ \mu+\frac{\alpha_3+\alpha_4}{2} \end{bmatrix} = \begin{bmatrix} \frac{\alpha_1-\alpha_2}{2} \\ \frac{\alpha_1-\alpha_2}{2} \\ \frac{\alpha_2-\alpha_1}{2} \\ \frac{\alpha_2-\alpha_1}{2} \\ \frac{\alpha_3-\alpha_4}{2} \\ \frac{\alpha_3-\alpha_4}{2} \\ \frac{\alpha_4-\alpha_3}{2} \\ \frac{\alpha_4-\alpha_3}{2} \end{bmatrix}$$

$$\phi = \frac{(X\beta)^T(P-P_0)X\beta}{2\sigma^2} = \frac{(\alpha_1-\alpha_2)^2 + (\alpha_3-\alpha_4)^2}{2\sigma^2}$$




* Test procedure, Reject $H_0$ if $F > F_{0.05}(2, 4)$

As I showed the above, we see the test statistic is the same

$$F = \frac{2\left((\bar{y}_{1\cdot}-\bar{y}_{2\cdot})^2 + (\bar{y}_{3\cdot}-\bar{y}_{4\cdot})^2\right)}{\sum_i\sum_j (y_{ij}-\bar{y}_{i\cdot})^2}$$

and the distribution of $F$ is the same, $F \sim F\left(2, 4, \phi = \frac{(\alpha_1-\alpha_2)^2 + (\alpha_3-\alpha_4)^2}{2\sigma^2}\right)$.