(a) $y = X\beta + e$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_6 \\ y_7 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}}_{7\times 5} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix}$$

$r(X) = 4 < 5$

(b) NE : $(X^T X)\beta = X^T y$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 & 4 & 4 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 4 & 4 \\ 4 & 0 & 0 & 4 & 4 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} y_{\cdot} \\ y_1 + y_2 - y_3 - y_4 \\ y_1 - y_2 + y_3 - y_4 \\ y_1 + y_2 + y_3 + y_4 \\ y_1 + y_2 + y_3 + y_4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 7\beta_0 + \qquad\qquad\qquad 4\beta_3 + 4\beta_4 &= y_{\cdot} \\ 4\beta_1 \qquad\qquad &= y_1 + y_2 - y_3 - y_4 \\ 4\beta_2 \qquad &= y_1 - y_2 + y_3 - y_4 \\ 4\beta_0 \qquad\qquad\qquad + 4\beta_3 + 4\beta_4 &= y_1 + y_2 + y_3 + y_4 \\ 4\beta_0 \qquad\qquad\qquad + 4\beta_3 + 4\beta_4 &= y_1 + y_2 + y_3 + y_4 \end{aligned}$$

(c) By Def, $AGA = A \Rightarrow G$ is a g-inverse of A

$$(X^TX)(X^TX)^-(X^TX) = \begin{bmatrix} 7 & 0 & 0 & 4 & 4 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 4 & 4 \\ 4 & 0 & 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & \frac{7}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 & 4 & 4 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 4 & 4 \\ 4 & 0 & 0 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 & 4 & 4 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 4 & 4 \\ 4 & 0 & 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 & 4 & 4 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 4 & 4 \\ 4 & 0 & 0 & 4 & 4 \end{bmatrix}$$

$$= (X^TX)$$

$\Rightarrow$ The given matrix is a g-inverse of (X^TX)

LSE of $\beta \Rightarrow \quad \tilde{\beta} = (X^TX)^- X^Ty + (I - (X^TX)^- X^TX)z, \quad z \in \mathbb{R}^5$

One solution is $\hat{\beta} = (X^TX)^- X^Ty$

$$\Rightarrow \hat{\beta} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & \frac{7}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_\cdot \\ y_1+y_2-y_3-y_4 \\ y_1-y_2+y_3-y_4 \\ y_1+y_2+y_3+y_4 \\ y_1+y_2+y_3+y_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(y_5+y_6+y_7) \\ \frac{1}{4}(y_1+y_2-y_3-y_4) \\ \frac{1}{4}(y_1-y_2+y_3-y_4) \\ \frac{1}{4}(y_1+y_2+y_3+y_4)-\frac{1}{3}(y_5+y_6+y_7) \\ 0 \end{bmatrix}$$

(d) $$P = X(X^TX)^-X^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & \frac{7}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} X^T$$

$$= \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\hat{y} = Py = \begin{bmatrix} \frac{3}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 - \frac{1}{4}y_4 \\ \frac{1}{4}y_1 + \frac{3}{4}y_2 - \frac{1}{4}y_3 + \frac{1}{4}y_4 \\ \frac{1}{4}y_1 - \frac{1}{4}y_2 + \frac{3}{4}y_3 + \frac{1}{4}y_4 \\ -\frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 + \frac{3}{4}y_4 \\ \frac{1}{3}y_5 + \frac{1}{3}y_6 + \frac{1}{3}y_7 \\ \frac{1}{3}y_5 + \frac{1}{3}y_6 + \frac{1}{3}y_7 \\ \frac{1}{3}y_5 + \frac{1}{3}y_6 + \frac{1}{3}y_7 \end{bmatrix}$$

(e) $\dim(N(X)) = 5 - \underbrace{\text{rank}(X)}_{=\mathcal{C}(X^T)} = 5 - 4 = 1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \text{ basis for } N(X)$$

(f) $\lambda^T\beta$ is estimable $\Rightarrow$ $\lambda \in \mathcal{C}(X^T)$

I will find a basis for $\mathcal{C}(X^T)$ $\Rightarrow$ The vectors in the basis are r linearly independent estimable functions.

A basis for $\mathcal{C}(X^T)$ is

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}_{\substack{=u_1 \\ =\lambda_1}}, \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\substack{=u_2 \\ =\lambda_2}}, \underbrace{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}_{\substack{=u_3 \\ =\lambda_3}}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\substack{=u_4 \\ =\lambda_4}}$$

4 linearly independent estimable functions. & $\lambda^T\hat{\beta}$ is the LSE of $\lambda^T\beta$ — not asked

$\lambda_1^T\beta = \beta_0 + \beta_1 - \beta_2 + \beta_3 + \beta_4 \Rightarrow \frac{1}{4}y_1 + \frac{3}{4}y_2 - \frac{1}{4}y_3 + \frac{1}{4}y_4 = \lambda_1^T\hat{\beta}$

$\lambda_2^T\beta = \beta_0 - \beta_1 + \beta_2 + \beta_3 + \beta_4 \Rightarrow \frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{3}{4}y_3 + \frac{1}{4}y_4 = \lambda_2^T\hat{\beta}$

$\lambda_3^T\beta = \beta_0 - \beta_1 - \beta_2 + \beta_3 + \beta_4 \Rightarrow -\frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 + \frac{3}{4}y_4 = \lambda_3^T\hat{\beta}$

$\lambda_4^T\beta = \beta_0 \Rightarrow \frac{1}{3}(y_5 + y_6 + y_7) = \lambda_4^T\hat{\beta}$

h – (i) $\beta_3 + \beta_4 = \underbrace{[0\ 0\ 0\ 1\ 1]}_{=\lambda^T}\beta$

$\lambda^T V = 0 \Rightarrow \lambda^T \perp N(X) \Rightarrow \lambda^T \in C(X^T)$

$\Rightarrow \beta_3 + \beta_4$ are estimable

(g) A condition $c \notin C(X^T)$ & since $r(X) = 4 \Rightarrow$ need one constraint.

So, one easy choice is $V = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ which implies $\beta_3 = \beta_4$.

$\Rightarrow V^T\beta = 0$

$$\Rightarrow \hat{\beta}_0 = \frac{y_5 + y_6 + y_7}{3}$$

$$\hat{\beta}_1 = \frac{y_1 + y_2 - y_3 - y_4}{4}$$

$$\hat{\beta}_2 = \frac{y_1 - y_2 + y_3 - y_4}{4}$$

$$\hat{\beta}_3 = \hat{\beta}_4 = \frac{y_1 + y_2 + y_3 + y_4}{8} - \frac{y_5 + y_6 + y_7}{6}$$

h-(i) is on the previous page!

h - (ii) $\beta_3+\beta_4 = [0\ 0\ 0\ 1\ 1]\beta$

$\Rightarrow$ LSE of $\lambda^T\beta = \lambda^T\hat{\beta} = \hat{\beta}_3 + \hat{\beta}_4$.

$= \frac{1}{4}(y_1+y_2+y_3+y_4) - \frac{1}{3}(y_5+y_6+y_7)$

h-(iii) $E(\lambda^T\hat{\beta}) = E\left(\frac{1}{4}(y_1+y_2+y_3+y_4) - \frac{1}{3}(y_5+y_6+y_7) \right)$

$= \frac{1}{4}\left(4\beta_0 + 4\beta_3 + 4\beta_4\right) - \frac{1}{3}(3\beta_0) = \beta_3+\beta_4 \Rightarrow$ unbiased

$$Var(\lambda^T\hat{\beta}) = \sigma^2\lambda^T(X^TX)^{-}\lambda = \sigma^2 [0\ 0\ 0\ 1\ 1] \begin{bmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & \frac{7}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \sigma^2 \left[-\frac{1}{3}\ 0\ 0\ \frac{7}{12}\ 0\right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{7}{12}\sigma^2$$

(i) From (c), $\hat{y}_1 = \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3 + \hat{\beta}_4$

$$= \frac{1}{3}(y_5 + \cancel{y_6} + y_7) + \frac{1}{4}(y_1 + \cancel{y_2} - \cancel{y_3} - \cancel{y_4}) + \frac{1}{4}(y_1 - \cancel{y_2} + \cancel{y_3} - y_4)$$

$$+ \frac{1}{4}(y_1 + y_2 + y_3 + \cancel{y_4}) - \frac{1}{3}(y_5 + \cancel{y_6} + y_7) + 0$$

$$= \frac{3}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 - \frac{1}{4}y_4$$

From (d), I have $\hat{y}_1 = \frac{3}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 - \frac{1}{4}y_4$

From (g), I have

$$\hat{y}_1 = \frac{y_5 + y_6 + y_7}{3} + \frac{1}{4}(y_1 + y_2 - y_3 - y_4) + \frac{1}{4}(y_1 - y_2 + y_3 - y_4)$$

$$+ 2\left(\frac{y_1 + y_2 + y_3 + y_4}{8} - \frac{y_5 + y_6 + y_7}{6}\right)$$

$$= \frac{3}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 - \frac{1}{4}y_4$$

Any solution to NEs, $(X^TX)\beta = X^Ty$ yields the same $\hat{y} = Py$.

2. (i) $E(d^Ty) = d^TX\beta = 0 \Rightarrow d^TX = 0$.

$$Cov(\lambda^T\hat{\beta}, d^Ty) = cov(\lambda^T(X^TX)^- X^Ty, d^Ty)$$

$$= \sigma^2\lambda^T(X^TX)^- \underbrace{X^Td}_{=0}$$

$$= 0$$

(ii) $E(d^T\hat{e}) = d^TE(\hat{e}) = 0 \Rightarrow d^T\hat{e}$ is an unbiased estimator of zero.

$\Rightarrow$ By the result in (i), $Cov(\lambda^T\hat{\beta}, d^T\hat{e}) = 0$