AMS 241: Bayesian Nonparametric Methods – Fall 2015
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Stochastic processes: basic concepts and definitions

Consider a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the **sample space** of the experiment, an **index set** T, and a **state space** S. A **stochastic process** is a collection

$$\mathcal{X} = \{X(\omega, t) : \omega \in \Omega, t \in T\}$$

such that:
(1) For any n and any set of index points $t_i \in T$, $i = 1,...,n$, $(X_{t_1}, ..., X_{t_n})$ is an n-dimensional random variable (random vector) defined on the probability space $(\Omega, \mathcal{F}, P)$ and taking values in $S^n \equiv S \times ... \times S$. (Hence, for each fixed $t_i \in T$, $X_{t_i}(\cdot) \equiv X(\cdot, t_i) : (\Omega, \mathcal{F}, P) \to S$ is a random variable.)
(2) For any fixed $\omega \in \Omega$, $X_\omega(\cdot) \equiv X(\omega, \cdot) : T \to S$ is a function defined on T and taking values in S, referred to as a **sample** (or **sample path** or **realization**) of the stochastic process $\mathcal{X}$.

Conditions (1) and (2) indicate that a stochastic process $\mathcal{X}$ can be viewed either as a collection of random variables $\{X_t : t \in T\}$ or as a collection of random functions $\{X_\omega : \omega \in \Omega\}$.

Depending on the nature of T and S, we can have discrete-time or continuous-time stochastic processes (countable or uncountable T, respectively) and discrete-state or continuous-state stochastic processes (countable or uncountable S, respectively).
For the details below, assume that S is a (countable or uncountable) subset of $\mathbb{R}^d$, $d \geq 1$ (the definitions can be extended to stochastic processes taking values in the complex plane).

Conditions (1) and (2) also indicate that for the study of a stochastic process both distributional properties and properties of sample paths are important. With regard to the former, the distribution function of the random vector $(X_{t_1}, ..., X_{t_n})$,

$$F_{\boldsymbol{t}}(x_1, ..., x_n) = \Pr(X_{t_1} \leq x_1, ..., X_{t_n} \leq x_n),$$

contains all the information for the specific index points $\boldsymbol{t} = (t_1, ..., t_n)$. The collection of all these distribution functions $F_{\boldsymbol{t}}$, as $\boldsymbol{t}$ ranges over all possible vectors of index points of any (finite) length n, is the set of **finite-dimensional distributions** (*fdds*) of the stochastic process $\mathcal{X}$.

The **Kolmogorov consistency conditions** ensure existence of a stochastic process associated with a set of fdds. Formally, assume that for each (finite) n and for each set of index points $\boldsymbol{t} = (t_1, ..., t_n)$ (in some index set T), we define a distribution function $F_{\boldsymbol{t}}$. If the collection of all such distribution functions satisfies the Kolmogorov consistency conditions:
(a) $F_{(t_1,...,t_n,t_{n+1})}(x_1, ..., x_n, x_{n+1}) \to F_{(t_1,...,t_n)}(x_1, ..., x_n)$ as $x_{n+1} \to \infty$, and
(b) For all n, $\boldsymbol{x} = (x_1, ..., x_n)$, $\boldsymbol{t} = (t_1, ..., t_n)$, and any permutation $\pi = (\pi(1), ..., \pi(n))$ of $\{1, 2, ..., n\}$, $F_{\pi\boldsymbol{t}}(\pi\boldsymbol{x}) = F_{\boldsymbol{t}}(\boldsymbol{x})$, where $\pi\boldsymbol{x} = (x_{\pi(1)}, ..., x_{\pi(n)})$ and $\pi\boldsymbol{t} = (t_{\pi(1)}, ..., t_{\pi(n)})$,
then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a collection $\mathcal{X} = \{X_t : t \in T\}$ of random variables, defined on $(\Omega, \mathcal{F}, P)$, such that the set of $F_{\boldsymbol{t}}$ is the set of fdds of $\mathcal{X}$.

It is important to note that fdds do not characterize a stochastic process, that is, they do not always yield complete information about properties of sample paths. It is possible to have two (or more) stochastic processes with the same set of fdds but with different sample paths. Such processes are called *versions* of one another. (Under conditions on the stochastic process $\mathcal{X}$, it can be shown that there exists a version $\mathcal{Y}$ of $\mathcal{X}$ with some specific property satisfied by its sample paths, e.g., right-continuity or differentiability.)

Using the information provided by the set of fdds, we can define several useful functions for a stochastic process $\mathcal{X}$. (For all the definitions below, we assume that the required expectations exist.) For any $t \in T$, the **mean function** of $\mathcal{X}$ is

$$\mu(t) \equiv \mathrm{E}(X_t) = \int x \, \mathrm{d}F_t(x).$$

For any $t_i, t_j \in T$, the **covariance function** is given by

$$c(t_i, t_j) \equiv \mathrm{Cov}(X_{t_i}, X_{t_j}) = \mathrm{E}(X_{t_i} X_{t_j}) - \mu(t_i)\mu(t_j)$$

and the **correlation function** by

$$r(t_i, t_j) \equiv \mathrm{Corr}(X_{t_i}, X_{t_j}) = \frac{\mathrm{Cov}(X_{t_i}, X_{t_j})}{\sqrt{\mathrm{Var}(X_{t_i})\mathrm{Var}(X_{t_j})}},$$

provided $\mathrm{Var}(X_{t_i}) > 0$ and $\mathrm{Var}(X_{t_j}) > 0$.

An important property of the autocovariance function is that it is a non-negative definite function, that is, $\sum_{i=1}^{k} \sum_{j=1}^{k} z_i z_j c(t_i, t_j) \geq 0$, for all (finite) k and for any t_1,...,$t_k \in T$ and real constants z_1,...,z_k.

If $c(t_i, t_j) = 0$, for all t_i, t_j with $t_i \neq t_j$, then the stochastic process $\mathcal{X}$ is typically called a **white noise** process. (If X_{t_i} and X_{t_j} are independent for all t_i, t_j with $t_i \neq t_j$, $\mathcal{X}$ is sometimes called a strictly white noise process.)

We say that $\mathcal{X}$ is a stochastic process with uncorrelated (orthogonal) increments if for any $t_i < t_j < t_k < t_l \in T$, $\mathrm{Cov}(X_{t_j} - X_{t_i}, X_{t_l} - X_{t_k}) = 0$ $(\mathrm{E}((X_{t_j} - X_{t_i})(X_{t_l} - X_{t_k})) = 0)$.

The process $\mathcal{X}$ has independent increments if for any $t_i < t_j < t_k < t_l \in T$, $X_{t_j} - X_{t_i}$ and $X_{t_l} - X_{t_k}$ are independent.